# National School SIDRA 2017: Formal Methods for the Control of Large-scale Networked Nonlinear Systems with Logic Specifications

## Lecture L2 & L9: Notes on Nonlinear Delay-Free and Retarded Systems Theory*

**Abstract.** In this lecture we will review the notions of forward completeness, global asymptotic stability and input-to-state stability, as well as their incremental versions, for nonlinear delay-free and time-delay systems.

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# 1 Notation, Standards, Acronyms

$\mathbb{R}$ stands for the real line $(-\infty, +\infty)$, $\mathbb{R}_0^+$ stands for the non-negative real line $[0, +\infty)$, $\mathbb{R}^\star$ stands for the extended real line $[-\infty, +\infty]$. $\mathbb{Q}$ stands for the set of rational numbers. $\mathbb{Z}$ stands for the set of the integer numbers, $\mathbb{Z}^+$ for the set of the integer numbers contained in $\mathbb{R}_0^+$. For a positive integer $n$, $\mathbb{R}^n$ is the linear space of $n-reals$ vectors. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. For $y \in \mathbb{R}^n$ and a positive real $r$, $B_r^n(y)$ denotes the subset of vectors $x \in \mathbb{R}^n$ such that $|x-y| \leq r$. The symbol $B_r^n$ is used instead of $B_r^n(0)$. A Lebesgue measurable function $u : [0, +\infty) \to \mathbb{R}^m$, $m$ positive integer, is said to be essentially bounded if $ess\sup_{t\geq 0} |u(t)| < +\infty$, where

$$ess\sup_{t\geq 0} |u(t)| = \inf\{a \in [0, +\infty] : \lambda(\{t \in [0, +\infty) : |u(t)| > a\}) = 0\},$$

$\lambda$ denoting the Lebesgue measure. The symbol $\|\cdot\|_\infty$ denotes the essential supremum norm, that is, for a Lebesgue measurable and essentially bounded function $u : [0, +\infty) \to \mathbb{R}^m$, $\|u\|_\infty = ess\sup_{t\geq 0} |u(t)|$. For given times $0 \leq T_1 < T_2$, we indicate with $u_{[T_1,T_2)} : [0, +\infty) \to \mathbb{R}^m$ the function given by $u_{[T_1,T_2)}(t) = u(t)$ for all $t \in [T_1, T_2)$ and $= 0$ elsewhere. An input $u$ is said to be *locally essentially bounded* if, for any $T > 0$, $u_{[0,T)}$ is essentially bounded. A function $\omega : \mathbb{R}^+ \to \mathbb{R}_0^+$ is said to be: increasing if $\omega(t_1) \leq \omega(t_2)$ $\forall t_1, t_2 \in \mathbb{R}_0^+$ such that $t_1 < t_2$; strictly increasing if $\omega(t_1) < \omega(t_2)$ $\forall t_1, t_2 \in \mathbb{R}_0^+$ such that $t_1 < t_2$; decreasing if $\omega(t_1) \geq \omega(t_2)$ $\forall t_1, t_2 \in \mathbb{R}_0^+$ such that $t_1 < t_2$; strictly decreasing if $\omega(t_1) > \omega(t_2)$ $\forall t_1, t_2 \in \mathbb{R}_0^+$ such that $t_1 < t_2$; of class $\mathcal{P}_0$ if it is continuous and satisfies $\omega(0) = 0$; of class $\mathcal{P}$ if it is of class $\mathcal{P}_0$ and $\omega(s) > 0$ holds for all $s > 0$; of class $\mathcal{G}$ if it is of class $\mathcal{P}_0$ and increasing; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded; of class $\mathcal{L}$ if it is continuous, decreasing and goes to zero as its argument goes to $+\infty$. A function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be of class $\mathcal{KL}$ if for each fixed $t$ the function $s \to \beta(s, t)$ is of class $\mathcal{K}$ and for each fixed $s$ the function $t \to \beta(s, t)$ is of class $\mathcal{L}$. For given (maximum involved time-delay) $\Delta > 0$, $n$ positive integer, $\mathcal{C}$ denotes the space of continuous functions mapping the interval $[-\Delta, 0]$ into $\mathbb{R}^n$ and for $\phi \in \mathcal{C}$, $\|\phi\|_\infty = \sup_{-\Delta\leq\theta\leq 0} |\phi(\theta)|$. For a given positive real $H$, $\mathcal{C}_H$ denotes the subset of continuous functions $\phi$ mapping the interval $[-\Delta, 0]$ into $\mathbb{R}^n$ such that $\|\phi\|_\infty \leq H$. $M_a : \mathcal{C} \to \mathbb{R}_0^+$ is any continuous functional such that there exist $\mathcal{K}_\infty$ functions $\underline{\gamma}_a$ and $\overline{\gamma}_a$ such that the inequalities hold

$$\underline{\gamma}_a(|\phi(0)|) \leq M_a(\phi) \leq \overline{\gamma}_a(\|\phi\|_\infty), \ \forall \phi \in \mathcal{C} \tag{1}$$

For any continuous function $x(s)$, defined on $-\Delta \leq s < a$, $a > 0$, and any fixed $t$, $0 \leq t < a$, the standard symbol $x_t$ will denote the element of $\mathcal{C}$ defined by $x_t(\theta) = x(t + \theta)$, $-\Delta \leq \theta \leq 0$. A function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be Lipschitz on bounded sets (equivalently, locally Lipschitz) if for any positive reals $H, \delta$ there exists a positive real $L_{H,\delta}$ such that, for any $x, y \in B_H^n$ and for any $u, v \in B_\delta^m$ , the inequality holds

$$|f(x, u) - f(y, v)| \leq L_{H,\delta}(|x - y| + |u - v|)$$

A map $f : \mathbb{C} \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be Lipschitz on bounded sets if for any positive reals $H, \delta$ there exists a positive real $L_{H,\delta}$ such that, for any $\phi, \psi \in \mathcal{C}_H$, and for any $u, v \in B^m_\delta$, the inequality holds

$$|f(x,u) - f(y,v)| \leq L_{H,\delta}(|x-y| + |u-v|)$$

A map $V : \mathcal{C} \to \mathbb{R}^+_0$ is said to be locally Lipschitz if, for any $\phi \in \mathcal{C}$ there exist positive reals $\delta_\phi$ and $L_\phi$ such that the following inequality holds

$$|V(\phi_1) - V(\phi_2)| \leq L_\phi \|\phi_1 - \phi_2\|_\infty, \qquad \forall \phi_1, \phi_2 \in \mathcal{C}_{\delta_\phi}(\phi). \tag{2}$$

In the following: ODE stands for ordinary differential equation; RFDE stands for retarded functional differential equation; GAS stands for global asymptotic stability or globally asymptotically stable; ISS stands for input-to-state stability or input-to-state stable; $\delta$-GAS stands for incremental global asymptotic stability or incrementally globally asymptotically stable; $\delta$-ISS stands for incremental input-to-state stability or incrementally input-to-state stable; $FC$ stands for forward completeness or forward complete; LM stands for Lebesgue measurable; LMEB stands for Lebesgue measurable essentially bounded; LMLEB stands for Lebesgue measurable locally essentially bounded.

## 2 Nonlinear Delay-Free Systems

In this section we will review some basic notions of nonlinear finite dimensionals systems, starting with the main hypothesis for the existence and uniqueness of solution. We will provide definitions of basic internal and external stability notions and related Lyapunov theorems.

Let us consider a nonlinear system described by the ODE ([11], [15])

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t), u(t)), & \text{a.e.} \\ x(0) = x_0, \end{cases} \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $n, m$ are positive integers, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz function satisfying $f(0,0) = 0$. The input signal $u : \mathbb{R}^+_0 \to \mathbb{R}^m$ is assumed to be Lebesgue measurable and locally essentially bounded.

The following theorem provides basic results concerning existence, uniqueness and continuity of the solution, for the system $\Sigma$.

**Theorem 1.** *([11], [15]) For any initial state $x_0 \in \mathbb{R}^n$, for any LMLEB input signal $u : \mathbb{R}^+_0 \to \mathbb{R}^m$:*

- *there exist, uniquely, a solution $x(t, x_0, u)$ on a maximal time interval $[0, b)$, with $0 < b \leq +\infty$, and such solution is locally absolutely continuous in $[0, b)$;*
- *if $b < +\infty$, then the solution is unbounded in $[0, b)$;*
- *$\forall \epsilon > 0$, and $\forall T \in (0, b)$, there exists a positive real $\delta$ such that, for any $y_0 \in \mathbb{R}^n$, and for any LMLEB input signal $v : \mathbb{R}^+_0 \to \mathbb{R}^m$ satisfying*

$$|x_0 - y_0| \leq \delta, \ |u(t) - v(t)| \leq \delta \ \text{a.e.},$$

*the solution $x(t, y_0, v)$ corresponding to $y_0, v$ exists in $[0, T]$ and, furthermore, the inequality holds:*

$$|x(t, x_0, u) - x(t, y_0, v)| \leq \epsilon, \quad \forall t \in [0, T]$$

Notice that, even if $b = +\infty$, the continuity of the solution with respect to initial states and input signals concerns a compact set.

*Example 1.* Let, in $\Sigma$, $n = m = 1$, $f(x, u) = u$, and let $u(t) = D(t)$, where $D : \mathbb{R}_0^+ \to \{0, 1\}$ is the Dirichlet function defined, for $t \in \mathbb{R}_0^+$ as

$$D(t) = \begin{cases} 1, \, t \in \mathbb{Q}, \\ 0, \; t \notin \mathbb{Q} \end{cases} \tag{4}$$

Then, the solution of $\Sigma$ is $x(t, x_0, u) = x_0, \;\; \forall t \geq 0$. Indeed, $x(0, x_0, u) = x_0$, and the derivative $\frac{dx(t,x_0,u)}{dt} = 0$, for all $t \geq 0$, thus satisfying the ODE in $\Sigma$ almost everywhere (recall that $\mathbb{Q}$ is a set of zero Lebesgue measure).

*Example 2.* Let, in $\Sigma$, $n = m = 1$, $f(x, u) = x^2 + u$, and let $u(\cdot) \equiv 0$. Then, the solution of $\Sigma$ is

$$x(t, x_0, 0) = \begin{cases} \frac{x_0}{1-x_0 t}, \;\; t \in [0, +\infty)\,, \, x_0 \leq 0, \\ \frac{x_0}{1-x_0 t}, \;\; t \in \left[0, \frac{1}{x_0}\right), \;\; x_0 > 0 \end{cases} \tag{5}$$

Indeed, $x(0, x_0, u) = x_0$, and the derivative $\frac{dx(t,x_0,u)}{dt} = x^2(t, x_0, u)$, for all $t \in [0, +\infty)$ in the case $x_0 \leq 0$, and for all $t \in \left[0, \frac{1}{x_0}\right)$ in the case $x_0 > 0$. The solution is unbounded in the latter case.

*Example 3.* Let, in $\Sigma$, $n = m = 1$, $f(x, u) = -x + x^2 + u$, and let $u(\cdot) \equiv 0$. Let us consider three cases

1) $x(0) = 1$,
2) $x(0) = 1 + \delta$,
3) $x(0) = 1 - \delta$,

where $\delta$ is a positive real. In case (1), the solution is $x(t, x_0, u) = 1, \; \forall t \in [0, +\infty)$. In case (2), the solution increases to $+\infty$ in the interval of definition $[0, b)$, $0 < b \leq +\infty$. In case (3) the solution is defined in $\mathbb{R}_0^+$, and goes to 0 as the time $t$ goes to $+\infty$. Nevertheless, by Theorem 1, for any positive real $\epsilon$ and any positive real $T$, there exist a positive real $\delta$ such that the solutions corresponding to the cases (2) and (3), for this positive real $\delta$, exist in $[0, T]$, and in $[0, T]$ differ from the one of case (1) less than $\epsilon$.

### 2.1 Forward Completeness and Internal stability

**Definition 1.** *([2], [33]) Let, in $\Sigma$, the input signal $u$ take value in a convex compact subset $\mathcal{D}$ of $\mathbb{R}^m$, containing the origin. System $\Sigma$ is forward complete if for every initial state $x_0$ and every LM input signal $u : \mathbb{R}_0^+ \to \mathcal{D}$, the corresponding solution exists for all $t \in \mathbb{R}_0^+$.*

**Definition 2.** *([11], [15]) System $\Sigma$ with $u(\cdot) \equiv 0$ is 0-GAS if there exists a function $\beta$ of class $\mathcal{KL}$ such that, $\forall x_0 \in \mathbb{R}^n$, the corresponding solution of $\Sigma$ exists $\forall t \geq 0$, and, furthermore, satisfies the inequality*

$$|x(t)| \leq \beta(|x_0|, t)$$

**Definition 3.** *([1]) Let, in $\Sigma$, the input signal $u$ take value in a convex compact subset $\mathcal{D}$ of $\mathbb{R}^m$, containing the origin. System $\Sigma$ is $\delta$-GAS (with respect to $\mathcal{D}$) if there exists a function $\beta \in \mathcal{KL}$ such that for any initial states $x_0, y_0 \in \mathbb{R}^n$, and for any LM input signal $u : \mathbb{R}_0^+ \to \mathcal{D}$, the corresponding solutions $x(t, x_0, u)$ and $x(t, y_0, u)$ exist $\forall t \geq 0$, and, furthermore, satisfy the inequality*

$$|x(t, x_0, u) - x(t, y_0, u)| \leq \beta(|x_0 - y_0|, t)$$

**Theorem 2.** *([2]) System $\Sigma$ is forward complete (with respect to $\mathcal{D}$), if and only if there exist a smooth function $V : \mathbb{R}^n \to \mathbb{R}_0^+$, and functions $\alpha_1$, $\alpha_2$ of class $\mathcal{K}_\infty$ such that, for any $x \in \mathbb{R}^n$ and for any $u \in \mathcal{D}$, the following inequalities hold*

*1)* $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$
*2)* $\frac{\partial V(x)}{\partial x} f(x, u) \leq V(x)$

**Theorem 3.** *([18], [11], [15]) System $\Sigma$, with $u(\cdot) \equiv 0$, is 0-GAS if and only if there exist a smooth function $V : \mathbb{R}^n \to \mathbb{R}_0^+$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a function $\alpha_3 \in \mathcal{K}$ such that, $\forall x \in \mathbb{R}^n$, the following inequalities hold*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

$$\frac{\partial V(x)}{\partial x} f(x, 0) \leq -\alpha_3(|x|)$$

**Theorem 4.** *([1]) System $\Sigma$ is $\delta-GAS$ (with respect to $\mathcal{D}$) if and only if there exist a smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$, and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$ such that, for any $x, y \in \mathbb{R}^n$ and for any $u \in \mathcal{D}$, the inequalities hold*

$$\alpha_1(|x - y|) \leq V(x, y) \leq \alpha_2(|x - y|)$$

$$\frac{\partial V(x, y)}{\partial x} f(x, u) + \frac{\partial V(x, y)}{\partial y} f(y, u) \leq -\alpha_3(|x - y|))$$

**Theorem 5.** *([1], [6], [29]) System $\Sigma$ is $\delta - GAS$ (with respect to $\mathcal{D}$) if and only if there exist a smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a positive real $\alpha_3$ such that, for any $x, y \in \mathbb{R}^n$ and for any $u \in \mathcal{D}$, the inequalities hold*

$$\alpha_1(|x - y|) \le V(x, y) \le \alpha_2(|x - y|)$$
$$\frac{\partial V(x, y)}{\partial x} f(x, u) + \frac{\partial V(x, y)}{\partial y} f(y, u) \le -\alpha_3 V(x, y)$$

### 2.2 External Stability

**Definition 4.** *([31], [11], [15]) System $\Sigma$ is ISS if there exist a function $\beta \in \mathcal{KL}$ and a function $\gamma \in \mathcal{K}$, such that for any initial state $x_0 \in \mathbb{R}^n$, for any LMLEB input signal $u : \mathbb{R}_0^+ \to \mathbb{R}^m$, the corresponding solution exists $\forall t \ge 0$, and, furthermore, satisfies the inequality*

$$|x(t)| \le \beta(|x_0|, t) + \gamma(ess \sup_{\theta \in [0,t)} |u(\theta)|)$$

**Definition 5.** *([1]) Let, in $\Sigma$, the input signal $u$ take value in a convex compact subset $\mathcal{D}$ of $\mathbb{R}^m$, containing the origin. System $\Sigma$ is $\delta$-ISS (with respect to $\mathcal{D}$) if there exist a function $\beta \in \mathcal{KL}$ and a function $\gamma \in \mathcal{K}$, such that for any initial states $x_0, y_0 \in \mathbb{R}^n$, and for any LM input signals $u, v : \mathbb{R}_0^+ \to \mathcal{D}$, the corresponding solutions $x(t, x_0, u)$ and $x(t, y_0, v)$ exist $\forall t \ge 0$, and, furthermore, satisfy the inequality*

$$|x(t, x_0, u) - x(t, y_0, v)| \le \beta(|x_0 - y_0|, t) + \gamma(ess \sup_{\theta \in [0,t)} |u(\theta) - v(\theta)|)$$

*Remark 1.* The $\delta$-ISS property implies the ISS property. For, just consider $y_0 = 0$ and $v(\cdot) \equiv 0$. The ISS property implies the $\delta$-GAS property. The $\delta$-GAS property implies the 0-GAS property.

*Example 4.* ([1]) Let, in $\Sigma$, $f(x, u) = -x + u^3$. Then, system $\Sigma$ is ISS but is not $\delta$-ISS.

*Example 5.* Let, in $\Sigma$, $f(x, u) = -x + x^3 + u$. Then, system $\Sigma$ is not 0-GAS, and thus not $\delta$-GAS, not ISS, not $\delta$-ISS.

**Theorem 6.** *([32], [11]) System $\Sigma$ is ISS if and only if there exist a smooth function $V : \mathbb{R}^n \to \mathbb{R}_0^+$, and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ such that, for any $x \in \mathbb{R}^n$ and for any $u \in \mathbb{R}^m$, the inequalities hold*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$$
$$\frac{\partial V(x)}{\partial x} f(x, u) \le -\alpha_3(|x|) + \sigma(|u|)$$

**Theorem 7.** *([1]) System $\Sigma$ is $\delta$-ISS (with respect to $\mathcal{D}$) if and only if there exist a smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$, functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{P}$ such that, for all $x, y \in \mathbb{R}^n$, and for any $u, v \in \mathcal{D}$, the following conditions hold:*

$$\alpha_1(|x-y|) \leq V(x,y) \leq \alpha_2(|x-y|)$$

$$\alpha_3(|x-y|) \geq |u-v| \Rightarrow \frac{\partial V(x,y)}{\partial x} f(x,u) + \frac{\partial V(x,y)}{\partial y} f(y,v) \leq -\sigma(|x-y|)$$

*Example 6.* Let, in $\Sigma$, $n = m = 1$, $f(x,u) = -x - x^3 + u$. We prove that $\Sigma$ is $\delta$-ISS, by means of Theorem 7. Choose $V(x,y) = (x-y)^2$. Then, $\alpha_1(s) = \alpha_2(s) = s^2$, $s \in \mathbb{R}_0^+$. Taking into account that $sign(x^3 - y^3) = sign(x-y)$, by Young's inequality, we have

$$\begin{aligned}
&\frac{\partial V(x,y)}{\partial x} f(x,u) + \frac{\partial V(x,y)}{\partial y} f(y,v) = \\
&2(x-y)(-x-x^3+u) - 2(x-y)(-y-y^3+v) = \\
&-2(x-y)^2 - 2(x-y)(x^3-y^3) + 2(x-y)(u-v) \leq \\
&-2(x-y)^2 + (x-y)^2 + (u-v)^2 \leq -(x-y)^2 + (u-v)^2
\end{aligned} \tag{6}$$

Thus, we can choose $\alpha_3(s) = \frac{1}{\sqrt{2}} s$, $\sigma(s) = \frac{1}{2} s^2$.

## 3 Nonlinear Retarded Systems

A time-invariant RFDE is an equation of the type ([5], [7], [9], [11], [19], [17], [21])

$$\begin{aligned}
&\dot{x}(t) = f(x_t, u(t)), \qquad t \geq 0, \qquad a.e., \\
&x(\tau) = x_0(\tau), \qquad \tau \in [-\Delta, 0],
\end{aligned} \tag{7}$$

where $x(t) \in \mathbb{R}^n$ is the internal variable, $u(t) \in \mathbb{R}^m$ is the input function, for $t \geq 0$ $x_t : [-\Delta, 0] \to \mathbb{R}^n$ is the function in $\mathcal{C}$ given by $x_t(\tau) = x(t+\tau)$, $\Delta$ is a positive real (the maximum involved delay), $f$ is a map from $\mathcal{C} \times \mathbb{R}^m$ to $\mathbb{R}^n$, $x_0 \in \mathcal{C}$, $m, n$ are positive integers. The space $\mathcal{C}$ is the state space of the system described by (7). Multiple discrete non-commensurate as well as distributed time-delays can appear in (7). We assume that the map $f$ is Lipschitz on bounded sets. We assume that $f(0,0) = 0$, thus ensuring that $x(t) = 0$ is the solution corresponding to zero initial state and zero input (i.e., the trivial solution). Moreover, we assume that the input signal $u : \mathbb{R}_0^+ \to \mathbb{R}^m$ is Lebesgue measurable and locally essentially bounded. In the following, when the specification of the initial state and of the input is necessary, for clarity of presentation, $x(t, \phi, u)$ $(x_t(\phi, u))$ will denote the solution expressed in $\mathbb{R}^n$ (expressed in $\mathcal{C}$) at time $t$ corresponding to initial condition $\phi$ and input $u$. $x(t, \phi)$ $(x_t(\phi))$ will denote the solution expressed in $\mathbb{R}^n$ (expressed in $\mathcal{C}$) at time $t$ corresponding to initial condition $\phi$ and zero input.

### 3.1 Existence, Uniqueness and Continuity of the Solution

**Theorem 8.** *([9], [11], [14], [16], [19], [21]) The following results hold:*

1) *for any initial state $x_0 \in \mathcal{C}$ and any Lebesgue measurable and locally essentially bounded input function $u$, the RFDE (7) admits a unique, locally absolutely continuous, solution $x(t)$ on a maximal time interval $[0, b)$, $0 < b \le +\infty$;*
2) *if $b < +\infty$, then the solution is unbounded in $[0, b)$;*
3) *for any $T \in (0, b)$ and any $\epsilon > 0$ there exist positive reals $\delta, \gamma$ such that, for any $\phi \in \mathcal{C}_\delta(x_0)$ and any Lebesgue measurable input signal $v : [0, T] \to \mathbb{R}^m$, satisfying the inequality $\operatorname{ess\,sup}_{t\in[0,T]} |v(t) - u(t)| \le \gamma$, the solution $x(t, \phi, v)$ of (7) corresponding to initial state $\phi$ and input $v$ exists in $[0, T]$ and satisfies the inequality*

$$|x(t, \phi, v) - x(t)| < \epsilon, \qquad t \in [0, T] \tag{8}$$

### 3.2 Internal and External Stability Definitions by $\mathcal{KL}$ Functions

**Definition 6.** *([11], [21], [25]) The system described by (7), with $u(t) = 0$ $t \in \mathbb{R}_0^+$, is said to be 0-GAS if there exist a function $\beta$ of class $\mathcal{KL}$ such that, for any $x_0 \in \mathcal{C}$, the corresponding solution exists for all $t \ge 0$ and, furthermore, satisfies the inequality*

$$|x(t)| \le \beta(\|x_0\|_\infty, t), \ \forall t \ge 0 \tag{9}$$

**Definition 7.** *([31], [24]) The system described by (7) is said to be ISS if there exist a function $\beta$ of class $\mathcal{KL}$ and a function $\gamma$ of class $\mathcal{K}$ such that, for any initial condition $x_0 \in \mathcal{C}$ and any Lebesgue measurable, locally essentially bounded input $u$, the corresponding solution exists for all $t \ge 0$ and, furthermore, satisfies*

$$|x(t)| \le \beta(\|x_0\|_\infty, t) + \gamma(\|u_{[0,t)}\|_\infty), \qquad \forall t \ge 0. \tag{10}$$

*Remark 2.* The ISS property implies the 0-GAS property. Moreover, for any Lebesgue measurable, essentially bounded input $u$, the state is bounded, and is ultimately bounded by a class $\mathcal{K}$ function of $\|u_{[0,\infty)}\|_\infty$. Because of the time-invariant character of the system described by (7), if $u(t)$ converges to zero, so does $x(t)$. Indeed, taking into account the time-invariant property of the system described by (7), the following inequality holds

$$|x(t)| \le \beta\left(\beta(\|x_0\|_\infty, 0) + \gamma(\|u_{[0,\infty)\|_\infty}), \frac{t}{2}\right) + \gamma\left(\|u_{\left[\frac{t}{2},t\right)}\right), \qquad t \in \mathbb{R}_0^+ \tag{11}$$

**Definition 8.** *([1], [27]) The system described by (7) is said to be $\delta$-ISS if there exist a function $\beta$ of class $\mathcal{KL}$ and a function $\gamma$ of class $\mathcal{K}$ such that, for any couple of initial states $x_0$, $\xi_0 \in \mathcal{C}$ and any couple of Lebesgue measurable, locally essentially bounded inputs $u_1$, $u_2$, the solutions corresponding to $(x_0, u_1)$ and $(\xi_0, u_2)$ exist for all $t \ge 0$ and, furthermore, the following inequality holds*

$$|x(t, x_0, u_1) - x(t, \xi_0, u_2)| \le \beta(\|x_0 - \xi_0\|_\infty, t) + \gamma(\|(u_1 - u_2)_{[0,t)}\|_\infty), \qquad \forall t \ge 0 \tag{12}$$

**Definition 9.** *Let $V : \mathcal{C} \to \mathbb{R}_0^+$ be a continuous functional. The derivative $D^+V : \mathcal{C} \times \mathbb{R}^m \to \mathbb{R}^\star$ of the functional $V$ is defined, for $\phi \in \mathcal{C}$, $v \in \mathbb{R}^m$, as follows ([4], [11], [21], [22], [23], [24])*

$$D^+V(\phi, v) = \limsup_{h\to 0^+} \frac{1}{h}\left(V(\phi_h) - V(\phi)\right), \tag{13}$$

*where $\phi_h \in \mathcal{C}$ is given by*

$$\phi_h(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h], \\ \phi(0) + f(\phi, v)(h+s), & s \in (-h, 0] \end{cases} \tag{14}$$

### 3.3 Lyapunov-Krasovskii Theorem for the 0-GAS Property

**Theorem 9.** *([11], [12], [17], [25]) The system described by (7), with $u(t) = 0$, $t \geq 0$, is 0-GAS if and only if there exist a locally Lipschitz functional $V : \mathcal{C} \to \mathbb{R}_0^+$ and functions $\alpha_1$, $\alpha_2$ of class $\mathcal{K}_\infty$, $\alpha_3$ of class $\mathcal{K}$, such that, $\forall \phi \in \mathcal{C}$, the following inequalities hold:*
*i)* $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty)$;
*ii)* $D^+V(\phi, 0) \leq -\alpha_3(|\phi(0)|)$

### 3.4 Lyapunov-Krasovskii Theorem for the ISS Property

**Theorem 10.** *([11], [13], [21], [24], [26]) The system (7) is ISS if and only if there exist a locally Lipschitz functional $V : \mathcal{C} \to \mathbb{R}_0^+$, a functional $M_a$, functions $\alpha_1, \alpha_2, \alpha_3$ of class $\mathcal{K}_\infty$ and a function $\rho$ of class $\mathcal{K}$ such that:*
*i)* $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(M_a(\phi)),\ \forall \phi \in \mathcal{C}$;
*ii)* $D^+V(\phi, u) \leq -\alpha_3(M_a(\phi)) + \rho(|u|),\ \forall\ \phi \in \mathcal{C}, u \in \mathbb{R}^m$.

### 3.5 Lyapunov-Krasovskii Theorems for the δ-ISS Property

For a locally Lipschitz functional $V : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_0^+$, let the derivative in the Driver's form $D^+V : \mathcal{C} \times \mathcal{C} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^\star$ be defined, for $\phi_i \in \mathcal{C}$, $d_i \in \mathbb{R}^m$, $i = 1, 2$, as follows ([27]):

$$\begin{aligned} D^+V(\phi_1, \phi_2, d_1, d_2) &= \limsup\nolimits_{h\to 0^+} \tfrac{V(\phi_1^h, \phi_2^h) - V(\phi_1, \phi_2)}{h}, \\ \phi_i^h(s) &= \begin{cases} \phi_i(s+h), & s \in [-\Delta, -h), \\ \phi_i(0) + (s+h) f(\phi_i, d_i), & s \in [-h, 0], \end{cases} \end{aligned} \tag{15}$$

$\phi_i \in \mathcal{C}$, $d_i \in \mathbb{R}^m$, $i = 1, 2$.

**Theorem 11.** *([27]) Let there exist a locally Lipschitz functional $V : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_0^+$, a functional $M_a$, functions $\alpha_1, \alpha_2$ of class $\mathcal{K}_\infty$, functions $\alpha_3, \rho$ of class $\mathcal{K}$ such that* *i)* $\alpha_1(M_a(\phi_1 - \phi_2)) \leq V(\phi_1, \phi_2) \leq \alpha_2(M_a(\phi_1 - \phi_2)),\ \forall \phi_1, \phi_2 \in \mathcal{C}$*;*

*ii)* $D^+V(\phi_1, \phi_2, d_1, d_2) \leq -\alpha_3(M_a(\phi_1 - \phi_2)),$

$\forall\ \phi_1, \phi_2 \in \mathcal{C},\ d_1, d_2 \in \mathbb{R}^m :\ M_a(\phi_1 - \phi_2) > \rho(|d_1 - d_2|)$

*Then, the system (7) is* $\delta - ISS$*.*

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