# National School SIDRA 2017: Formal Methods for the Control of Large-scale Networked Nonlinear Systems with Logic Specifications

## Lecture L6a: Symbolic models for stable nonlinear systems*

**Abstract.** In this lecture we present some results for the construction of symbolic models for nonlinear systems. We will show that if the system is incrementally globally-asymptotically stable and the set of states and the set of inputs are bounded then it is possible to construct a symbolic model that approximates the original system in the sense of approximate bisimulation for any desired accuracy. This lecture is based on [4], see also [2, 6].

## 1 Notation

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of nonnegative integer, integer, real, positive real, and nonnegative real numbers, respectively. Given $a, b \in \mathbb{Z}$, we denote $[a; b] = [a, b] \cap \mathbb{Z}$. Given a pair of sets $X$ and $Y$ and a relation $\mathcal{R} \subseteq X \times Y$, the symbol $\mathcal{R}^{-1}$ denotes the inverse relation of $\mathcal{R}$, i.e. $\mathcal{R}^{-1} = \{(y, x) \in Y \times X : (x, y) \in \mathcal{R}\}$. Given $X' \subseteq X$ and $Y' \subseteq Y$, we denote $\mathcal{R}(X') = \{y \in Y | \exists x \in X' \text{ s.t. } (x, y) \in \mathcal{R}\}$ and $\mathcal{R}^{-1}(Y') = \{x \in X | \exists y \in Y' \text{ s.t. } (x, y) \in \mathcal{R}\}$. Given a function $f : X \to Y$ and $X' \subseteq X$ the symbol $f(X')$ denotes the image of $X'$ through $f$, i.e. $f(X') = \{y \in Y | \exists x \in X' \text{ s.t. } y = f(x)\}$. Given functions $f : X \to Y$ and $g : Y \to Z$ we denote by $g \circ f$ the composition of functions $f$ and $g$ that is the function $(g \circ f) : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; $\gamma$ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if for each fixed $s$, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to $r$ and, for each fixed $r$, the map $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to \infty$. Given a vector $x \in \mathbb{R}^n$ we denote by $x(i)$ the $i$–th element of $x$ and by $|x|$ the infinity norm of $x$. Given $a \in \mathbb{R}$ and $X \subseteq \mathbb{R}^n$, the symbol $aX$ denotes the set $\{y \in \mathbb{R}^n | \exists x \in X \text{ s.t. } y = ax\}$.

## 2 Symbolic models for nonlinear systems

Consider the following nonlinear control system:

$$\varSigma : \begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ x(t) \in \mathbf{X} = \mathbb{R}^n, \\ u(t) \in \mathbf{U} \subseteq \mathbb{R}^m, t \in \mathbb{R}_0^+, \end{cases}$$

where $x(t)$ is the state and $u(t)$ is the input at time $t \in \mathbb{R}_0^+$. Control inputs $u$ are assumed to belong to the class $\mathcal{U}$ of piecewise continuous functions from $\mathbb{R}_0^+$ to $\mathbf{U}$. For simplicity we assume that function $f$ is such that $\varSigma$ admits a unique solution for any initial state $x(0) \in \mathbf{X}$ and for any control input function $u \in \mathcal{U}$ and it is forward complete, i.e. starting from any initial state $x(0) \in \mathbf{X}$ and for any control input function $u \in \mathcal{U}$, the solution $\mathbf{x}(\cdot, x_0, u)$ to the differential equation $\varSigma$ exists for any time $t \in \mathbb{R}_0^+$. We also assume here that state variables are available for control purposes. We also assume that the use $\mathbf{U}$ is finite as it is often the case in concrete applications.
We have already represented $\varSigma$ as the transition system

$$T(\varSigma) = (X, X_0, U, \longrightarrow, X_m, Y, H),$$

where

- $X = \mathbf{X}$;
- $X_0 = \mathbf{X}_0$;

- $U$ is the collection of restrictions of functions in $\mathcal{U}$ to intervals $[0,\tau[$, for some $\tau \in \mathbb{R}^+$;
- $x \xrightarrow{u|_{[0,\tau[}} x'$ if $x' = \mathbf{x}(\tau, x, u)$;
- $X_m = \mathbf{X}$;
- $Y = \mathbf{Y}$ and
- $H(x) = h(x)$ for all $x \in X$.

Transition system above is metric when we regard $X \subseteq \mathbb{R}^n$ as equipped with a metric. In the sequel we use as metric, the infinite norm, i.e.

$$\mathbf{d}(x, x') = |x - x'|, \forall x, x' \in X.$$

The basic idea in deriving symbolic models for $\Sigma$ is to first proceed with a time discretization of the original system and then with a state space quantization.
We start with the time discretization of $\Sigma$.

**Definition 1.** *Given $\Sigma$ and a sampling time $\tau \in \mathbb{R}^+$, define*

$$T_\tau(\Sigma) = (X_\tau, X_{0,\tau}, U_\tau, \underset{\tau}{\longrightarrow}, X_{m,\tau}, Y_\tau, H_\tau),$$

*where*

- $X_\tau = X_{0,\tau} = X_{m,\tau} = \mathbf{X}$;
- $U_\tau$ *is the set of constant input functions* $u : [0,\tau[\to \mathbf{U}$;
- $x \underset{\tau}{\xrightarrow{u}} x'$ *if* $x' = \mathbf{x}(\tau, x, u)$;
- $Y_\tau = \mathbb{R}^n$ *and*
- $H_\tau(x) = x$ *for all* $x \in X_\tau$.

What are connections between $T(\Sigma)$ and $T_\tau(\Sigma)$?

- $T(\Sigma)$ and $T_\tau(\Sigma)$ have an infinite number of states;
- $T(\Sigma)$ and $T_\tau(\Sigma)$ are deterministic;
- $T(\Sigma)$ and $T_\tau(\Sigma)$ are alive;
- $T_\tau(\Sigma)$ is a subsystem of $T(\Sigma)$;
- $T(\Sigma)$ and $T_\tau(\Sigma)$ are metric.

We now proceed with the state space quantization of $T_\tau(\Sigma)$.
Given $\theta \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$, we denote

$$\mathcal{B}^n_{[-\theta,\theta[}(x) = \{y \in \mathbb{R}^n | y(i) \in [-\theta + x(i), \theta + x(i)[, i \in [1; n]\}.$$

A graphical representation of the set $\mathcal{B}^n_{[-\theta,\theta[}(x)$ in $\mathbb{R}^2$ is given in Fig. 1.
Note that for any $\theta \in \mathbb{R}^+$, the collection of sets $\mathcal{B}^n_{[-\theta,\theta[}(x)$ with $x$ ranging in $2\theta\,\mathbb{Z}^n$ is a partition of $\mathbb{R}^n$. We now define the quantization function.

**Definition 2.** *Given a quantization parameter $\theta \in \mathbb{R}^+$, the quantizer in $\mathbb{R}^n$ with accuracy $\theta$ is a function*

$$[\,\cdot\,]^n_\theta : \mathbb{R}^n \to 2\theta\mathbb{Z}^n,$$

*associating to any $x \in \mathbb{R}^n$ the unique vector $[x]^n_\theta \in 2\theta\mathbb{Z}^n$ such that $x \in \mathcal{B}^n_{[-\theta,\theta[}([x]^n_\theta)$.*

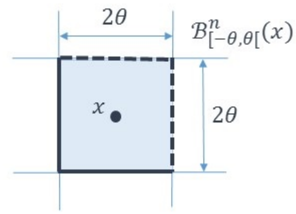


**Fig. 1.** Graphical representation of the set $\mathcal{B}^n_{[-\theta,\theta[}(x)$ in $\mathbb{R}^2$.

Definition of $[\,\cdot\,]^n_\theta$ naturally extends to sets $\Omega \subseteq \mathbb{R}^n$ when $[\Omega]^n_\theta$ is interpreted as the image of $\Omega$ through function $[\,\cdot\,]^n_\theta$.
We can now give the following

**Definition 3.** *Given $\Sigma$, a sampling time $\tau \in \mathbb{R}^+$ and a state space quantization $\eta \in \mathbb{R}^+$, define*

$$T_{\tau,\eta}(\Sigma) = (X_{\tau,\eta}, X_{0,\tau,\eta}, U_{\tau,\eta}, \underset{\tau,\eta}{\longrightarrow}, X_{m,\tau,\eta}, Y_{\tau,\eta}, H_{\tau,\eta}),$$

*where*

- $X_{\tau,\eta} = X_{0,\tau,\eta} = X_{m,\tau,\eta} = [\mathbf{X}]^n_\eta$;
- $U_{\tau,\eta}$ *is the set of constant input functions* $u : [0, \tau[ \to \mathbf{U}$;
- $\xi \overset{u}{\underset{\tau,\eta}{\longrightarrow}} \xi'$ *if* $\xi' = [\mathbf{x}(\tau, \xi, u)]^n_\eta$;
- $Y_{\tau,\eta} = \mathbb{R}^n$ *and*
- $H_{\tau,\eta}(x) = x$ *for all* $x \in X_{\tau,\eta}$.

The intuition behind this definition is to replace any state of $T_\tau(\Sigma)$ by its quantization. Transition system $T_{\tau,\eta}(\Sigma)$ is countable and becomes symbolic when the set $\mathbf{X}$ is bounded. In Fig. 2 we show a graphical representation of the construction of transition system $T_{\tau,\eta}(\Sigma)$.

What are relationships between $T_\tau(\Sigma)$ and $T_{\tau,\eta}(\Sigma)$?

- $T_\tau(\Sigma)$ has an infinite number of states while $T_{\tau,\eta}(\Sigma)$ has a countable number of states;
- $T_\tau(\Sigma)$ and $T_{\tau,\eta}(\Sigma)$ are deterministic;
- $T_\tau(\Sigma)$ and $T_{\tau,\eta}(\Sigma)$ are alive;
- $T_{\tau,\eta}(\Sigma)$ is not a subsystem of $T_\tau(\Sigma)$.

The results presented in this lecture will assume certain stability assumptions introduced in lecture L2 and briefly recalled hereafter.

**Definition 4.** *[1] A control system $\Sigma$ is incrementally globally asymptotically stable ($\delta$–GAS) if it is forward complete and there exist a $\mathcal{KL}$ function $\beta$ such that for any $t \in \mathbb{R}^+_0$, any $x, y \in \mathbb{R}^n$ and any $\mathbf{u} \in \mathcal{U}$ the following condition is satisfied:*

$$|\mathbf{x}(t, x, \mathbf{u}) - \mathbf{x}(t, y, \mathbf{u})| \le \beta(|x - y|, t). \tag{1}$$

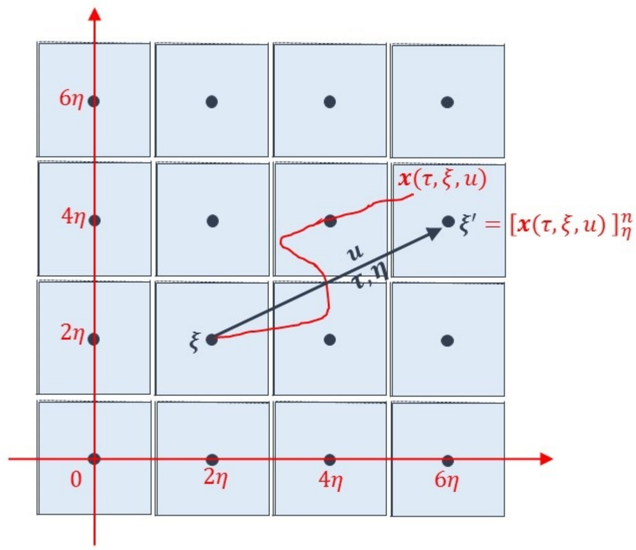


**Fig. 2.** Graphical representation of the transition system $T_{\tau,\eta}(\Sigma)$.

Definition above can be thought of as an incremental version of the classical notion of global asymptotic stability (GAS) [3]. In general, inequality (1) is difficult to check directly. Fortunately $\delta$–GAS can be characterized by dissipation inequalities.

**Definition 5.** *Consider a control system $\Sigma$ and a smooth function*

$$V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+.$$

*Function $V$ is called a $\delta$–GAS Lyapunov function for $\Sigma$, if there exist $\kappa \in \mathbb{R}^+$ and $\mathcal{K}_\infty$ functions $\alpha_1$ and $\alpha_2$ such that:*

*(i) for any $x, y \in \mathbb{R}^n$*

$$\alpha_1(|x - y|) \le V(x, y) \le \alpha_2(|x - y|);$$

*(ii) for any $x, y \in \mathbb{R}^n$ and any $u \in U$*

$$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial y} f(y, u) < -\kappa V(x, y).$$

*Remark 1.* In the classical formulation of $\delta$–GAS Lyapunov functions, condition (ii) in the definition above is replaced by:

(ii') for any $x, y \in \mathbb{R}^n$ and any $u \in U$

$$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial y} f(y, u) < -\alpha_3(|x - y|),$$

for some $\mathcal{K}$ function $\alpha_3$. However, it has been shown in [5] that there is no loss of generality in replacing (ii') by (ii).

The following result holds.

**Theorem 1.** *[1] Control system $\Sigma$ is $\delta$–GAS if it admits a $\delta$–GAS Lyapunov function.*

We also assume existence of a $\mathcal{K}_\infty$ function $\gamma$ such that

$$\forall x, y, z \in \mathbb{R}^n,\ |V(x,y) - V(x,z)| \leq \gamma(|y-z|). \tag{2}$$

Note that $\gamma$ is not a function of the variable $x$. This assumption is not restrictive provided that we are interested in the dynamics of $\Sigma$ on a compact subset of the state space $\mathbb{R}^n$.

We can now give the following result.

**Theorem 2.** *Consider control system $\Sigma$ and suppose it admits a $\delta$–GAS Lyapunov function $V$ and hence, satisfying conditions of Definition 5, for some $\kappa \in \mathbb{R}^+$ and $\mathcal{K}_\infty$ functions $\alpha_1$ and $\alpha_2$ and (2) for some $\mathcal{K}_\infty$ function $\gamma$. Then, for any desired accuracy $\mu \in \mathbb{R}^+$ and any sampling time $\tau \in \mathbb{R}^+$, select quantization parameter $\eta \in \mathbb{R}^+$ satisfying:*

$$\eta \leq \min\left\{\gamma^{-1}((1-e^{-\kappa\tau})\alpha_1(\mu)), (\alpha_2^{-1} \circ \alpha_1)(\mu)\right\}. \tag{3}$$

*Then, relation $\mathcal{R}_\mu \subseteq X_\tau \times X_{\tau,\eta}$ specified by*

$$(x,\xi) \in \mathcal{R}_\mu \Leftrightarrow V(x,\xi) \leq \alpha_1(\mu) \tag{4}$$

*is a $\mu$–approximate bisimulation relation between $T_\tau(\Sigma)$ and $T_{\tau,\eta}(\Sigma)$. Consequently, $T_\tau(\Sigma)$ and $T_{\tau,\eta}(\Sigma)$ are approximately bisimilar with accuracy $\mu$.*

*Proof.* We first show that $\mathcal{R}_\mu$ is a $\mu$–simulation relation from $T_\tau(\Sigma)$ to $T_{\tau,\eta}(\Sigma)$, i.e. it satisfies the following conditions:

i) $\forall x \in X_{0,\tau}\ \exists \xi \in X_{0,\tau,\eta}$ such that $(x,\xi) \in \mathcal{R}_\mu$;
ii) $\forall x \in X_{m,\tau}\ \exists \xi \in X_{m,\tau,\eta}$ such that $(x,\xi) \in \mathcal{R}_\mu$;
iii) $\forall (x,\xi) \in \mathcal{R}_\mu$,

$$\mathbf{d}(x,\xi) = |H_\tau(x) - H_{\tau,\eta}(\xi)| = |x-\xi| \leq \mu;$$

iv) $\forall (x,\xi) \in \mathcal{R}_\mu$ if $x \xrightarrow[\tau]{u} x'$ then there exists $\xi \xrightarrow[\tau,\eta]{u'} \xi'$ such that $(x',\xi') \in \mathcal{R}_\mu$.

Proof of i). For any $x \in X_{0,\tau}$ pick $\xi = [x]_\eta \in X_{0,\tau,\eta}$. We first note that

$$|x-\xi| \leq \eta.$$

Moreover

$$V(x,\xi) \leq \alpha_2(|x-\xi|) \leq \alpha_2(\eta) \leq \alpha_1(\mu),$$

where the last inequality holds by condition (3).
Proof of ii). Same reasoning as in the proof of i).
Proof of iii). Pick any $(x,\xi) \in \mathcal{R}_\mu$, i.e. such that $V(x,\xi) \leq \alpha_1(\mu)$. Then

$$\mathbf{d}(x,\xi) = |x-\xi| \leq \alpha_1^{-1}(V(x,\xi)) \leq \alpha_1^{-1}(\alpha_1(\mu)) = \mu.$$

Proof of iv). $\forall (x,\xi) \in \mathcal{R}_\mu$ consider any $x \xrightarrow[\tau]{u} x'$. Set $z = \mathbf{x}(\tau,\xi,u)$, $\xi' = [z]_\mu$ and consider the transition $\xi \xrightarrow[\tau,\eta]{u} \xi'$ in $T_{\tau,\eta}(\Sigma)$. We get:

$$\begin{aligned} V(x',\xi') &\leq V(x',z) + \gamma(|z-\xi'|) \\ &\leq V(\mathbf{x}(\tau,x,u),\mathbf{x}(\tau,\xi,u)) + \gamma(\eta) \\ &\leq e^{-\kappa\tau} V(x,\xi) + \gamma(\eta) \\ &\leq e^{-\kappa\tau}\alpha_1(\mu) + \gamma(\eta) \\ &\leq \alpha_1(\mu) \end{aligned}$$

where the last inequality holds by condition (3).
We now show that $\mathcal{R}_\mu^{-1}$ is a $\mu$–simulation relation from $T_{\tau,\eta}(\Sigma)$ to $T_\tau(\Sigma)$, i.e. it satisfies the following conditions:

i') $\forall \xi \in X_{0,\tau,\eta}$ $\exists x \in X_{0,\tau}$ such that $(\xi,x) \in \mathcal{R}_\mu^{-1}$;
ii') $\forall \xi \in X_{m,\tau,\eta}$ $\exists x \in X_{m,\tau}$ such that $(\xi,x) \in \mathcal{R}_\mu^{-1}$;
iii') $\forall (\xi,x) \in \mathcal{R}_\mu^{-1}$,

$$\mathbf{d}(\xi,x) = |H_{\tau,\eta}(\xi) - H_\tau(x)| = |\xi - x| \leq \mu;$$

iv') $\forall (\xi,x) \in \mathcal{R}_\mu^{-1}$ if $\xi \xrightarrow[\tau,\eta]{u'} \xi'$ then there exists $x \xrightarrow[\tau]{u} x'$ such that $(\xi',x') \in \mathcal{R}_\mu^{-1}$.

Proof of i'). For any $\xi \in X_{0,\tau,\eta}$ pick $x = \xi \in X_{0,\tau}$, from which

$$V(x,\xi) \leq \alpha_2(|x-\xi|) = 0 \leq \mu.$$

Proof of ii'). Same reasoning as in the proof of i').
Proof of iii'). Same reasoning as in the proof of iii).
Proof of iv'). Same reasoning as in the proof of iv).

The following counterexample shows that unstable control systems do not admit, in general, approximately bisimilar countable symbolic models.

*Example 1.* Consider the scalar autonomous linear system

$$\Sigma : \begin{cases} \dot{x}(t) = x, \\ x(t) \in \mathbf{X} = \mathbb{R}, \\ t \in \mathbb{R}_0^+. \end{cases}$$

System $\Sigma$ is unstable and hence not $\delta$–GAS. We now show that for any $\mu \in \mathbb{R}_0^+$, any $\tau \in \mathbb{R}^+$ and any countable transition system $T$, transition systems $T_\tau(\Sigma)$ and $T$ are *not* $\mu$–bisimilar. Consider any countable metric transition system

$$T = (X, X_0, U, \longrightarrow, X, X_m, \mathbb{R}, H),$$

with $H : X \to \mathbb{R}$ and the same metric $\mathbf{d}(x, x') = |x - x'|$ of $T_\tau(\Sigma)$. Consider any relation

$$\mathcal{R} \subseteq X \times X_\tau$$

satisfying the conditions of $\mu$–approximate bisimulation. In particular, since $X_{0,\tau} = X_\tau = \mathbf{X}$, by condition i) of Definition 5 of approximate simulation in lecture L5, we get:

$$\mathcal{R}(X_\tau) = X, \quad \mathcal{R}^{-1}(X) = X_\tau. \tag{5}$$

We now show that such relation $\mathcal{R}$ does not exist. By countability of $T$, there exist $\xi \in X$ and $x, x' \in X_\tau = \mathbb{R}$ such that $x \neq x'$, and $(x, \xi), (x', \xi) \in \mathcal{R}$. Set

$$\begin{aligned} x_k &= e^{\tau k} x, \\ x'_k &= e^{\tau k} x', \end{aligned}$$

for any $k \in \mathbb{N}$. Since $x \neq x'$, by selecting $\lambda \in \mathbb{R}^+$ such that $|x - x'| > \lambda$, we have:

$$|x_k - x'_k| = e^{\tau k}|x - x'| > e^{\tau k}\lambda, \forall k \in \mathbb{N}. \tag{6}$$

Choose $k' \in \mathbb{N}$ such that

$$e^{\tau k'}\lambda - \mu > \mu. \tag{7}$$

By Definition 6 (approximate bisimulation) and condition

iv) $\forall (x_1, x_2) \in \mathcal{R}$ if $x_1 \xrightarrow[1]{u_1} x'_1$ then there exists $x_2 \xrightarrow[2]{u_2} x'_2$ such that $(x'_1, x'_2) \in \mathcal{R}$.

of Definition 5 (approximate simulation) of lecture L5, and by (5), there must exist $\xi_{k'} \in X$ such that, $(x_{k'}, \xi_{k'}), (x'_{k'}, \xi_{k'}) \in \mathcal{R}$. Since $(x_{k'}, \xi_{k'}) \in \mathcal{R}$,

$$|x_{k'} - H(\xi_{k'})| \leq \mu. \tag{8}$$

By combining inequalities (6), (8) and (7), we obtain:

$$\begin{aligned} |H(\xi_{k'}) - x'_{k'}| &\geq |x_{k'} - x'_{k'}| - |x_{k'} - H(\xi_{k'})| \\ &> e^{\tau k'}\lambda - \mu > \mu. \end{aligned} \tag{9}$$

Inequality (9) shows that the pair $(x'_{k'}, \xi_{k'}) \in \mathcal{R}$ does not satisfy condition

iii') $\forall (x_1, x_2) \in \mathcal{R}$, $\mathbf{d}(H_1(x_1), H_2(x_2)) \leq \mu$.

of Definition 5 (approximate simulation) of lecture L5. Hence, there does not exist a $\mu$–approximate bisimulation relation between $T_\tau(\Sigma)$ and $T$ and consequently $T_\tau(\Sigma)$ and $T$ are *not* $\mu$–bisimilar.

## References


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